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Readers' Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "New Similarity Solutions for Hypersonic Boundary Layers with Application to Inlet Flows," Part 1

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IN this paper[1] two new self-similar solutions for two-dimensional laminar hypersonic boundary layers are derived to complement previous well-known solutions.[2] A feature of the new solutions is that the self-similar body shapes are preceded by a plain ramp, on which, presumably, a boundary layer develops upstream of the body, which generates a power law or exponential self-similar boundary layer. It is necessary at the junction between the ramp and the curved body that the boundary layer is continuous and hence that the boundary layer generated by the ramp is a solution of the downstream self-similar curved body boundary layer. That is, the distributions of density, velocity, and entropy across the boundary layer at the junction must be the same for both the ramp and curved body self-similar solutions.

The transformed y coordinate across the boundary layer is given by Inger's[1] Eq. (1) to be

$$\eta = \frac{U_e}{G}\int_0^y \rho \, \mathrm{d}y \tag{1}$$

where U_e is the velocity at the edge of the boundary layer and G is a function of x found as part of the solution. This y transformation can be made the same for both boundary-layer solutions at the junction, by choosing appropriate values of arbitrary constants that appear in the solutions for G, to make the value of G at the junction the same for both solutions.

Based on this common y coordinate transformation, the variation of the velocity and entropy across the boundary layer must be the same. For self-similar solutions, these are given by Inger's Eqs. (14) and (15) to be

$$ff'' + f''' = \lambda(1 - 1/\gamma)(g - f'^2) \tag{2}$$

$$fg' + g'' = 0 \tag{3}$$

where

$$\frac{u}{u_e} = f'(\eta) = \frac{\mathrm{d}f}{\mathrm{d}\eta} \tag{4}$$

$$H/H_e = g(\eta) \tag{5}$$

The boundary layer on the plain ramp can be described by the self-similar solution of Eqs. (2–5) by putting $\lambda = 0$ in Eq. (2). The boundary layer on the power law or exponential downstream body is described by $\lambda(1 - 1/\gamma)$ being a nonzero constant (for a curved body). The variation of f and g with η given by Eqs. (2) and (3) depends on the value of this constant and differs for different values of the constant. This is illustrated, for example, in Inger's Fig. 3, where a change in β_{CR} [$= -\lambda(1 - 1/\lambda)$] results in a change in the shear function at the wall.

Hence the variation of f and g with η at the junction where $\lambda(1 - 1/\lambda)$ changes value is not continuous for the two self-similar solutions, and the initial boundary layer at the start of the curved body will not have the correct self-similar form. Thus, certainly near the beginning of the curved self-similar region, the boundary-layer distribution and other dependent boundary-layer values must be expected to differ from the self-similar analytically derived expressions of Ref. 1.

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References

[1] Inger, G. R., "New Similarity Solutions for Hypersonic Boundary Layers with Application to Inlet Flows," *AIAA Journal*, Vol. 33, No. 11, 1995, pp. 2080–2086.

[2] Mirels, H., "Hypersonic Flow over Slender Bodies with Power Law Shocks," *Advances in Applied Mechanics VII*, Academic, New York, 1962, pp. 1–54.

Comment on "New Similarity Solutions for Hypersonic Boundary Layers with Application to Inlet Flows," Part 2

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IN the paper by Inger,[1] two new self-similar solutions to the two-dimensional laminar hypersonic boundary-layer equations are found, which complement the previous two well-known solutions.[2] All of these solutions are power law or exponential solutions valid for particular x coordinate transformations $\xi = \xi(x)$. Solution cases 1 and 3 of Inger's paper are valid when $\xi = x$ and cases 2 and 4 when ξ is a function of

$$\int p_e \, \mathrm{d}x$$

This separation into various cases[1] for various ξ is shown here to be unnecessary, for they are all examples of a general solution valid for all ξ.

To obtain solutions that are self-similar, the coefficients of the terms in $f(\eta)$ and $g(\eta)$ in Eqs. (5), (6), and (9) of Ref. 1 need to be constants. These equations can be written as

$$\left(\frac{\mathrm{d}G/\mathrm{d}\xi}{G}\right) ff'' + K_G f''' = \left(\frac{\gamma - 1}{2\gamma}\right)\frac{(\mathrm{d}p_e/\mathrm{d}\xi)}{p_e}(g - f'^2) \tag{1}$$

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$$\left(\frac{dG/d\xi}{G}\right)fg' + K_G g'' = 0 \tag{2}$$

where

$$K_G = C_w\rho_\infty\mu_\infty U_\infty\left(\frac{p_e}{p_\infty}\right)\bigg/\left(\frac{G^2 d\xi}{dx}\right) \tag{3}$$

with $C_w = \mu_w T_\infty/\mu_\infty T_w$, the Chapman–Rubesin constant.

Constant coefficients in Eqs. (1) and (2) require[1]

$$\frac{dp_e/d\xi}{2p_e} = \lambda\frac{dG/d\xi}{G} \tag{4}$$

and

$$K_G = \frac{dG/d\xi}{G} \tag{5}$$

Assuming that ξ is a differentiable function of x, Eqs. (3) and (5) give p_e as

$$p_e = \frac{p_\infty}{C_w\rho_\infty\mu_\infty U_\infty}\frac{G\,dG}{dx} \tag{6}$$

Multiplying Eq. (4) through by $d\xi/dx$ and substituting for p_e from Eq. (6) give an equation for $G(x)$ as

$$GG_{xx} = (2\lambda - 1)G_x^2 \tag{7}$$

This has the solution

$$\begin{aligned} G &= (A + Bx)^{\frac{1}{2}(1-\lambda)} \qquad \lambda \neq 1 \\ &= Ae^{Bx} \qquad \lambda = 1 \end{aligned} \tag{8}$$

with A and B as constants and

$$p_e = \frac{p_\infty}{C_\infty\rho_\infty\mu_\infty U_\infty}G^{2\lambda} \tag{9}$$

All of the solutions for cases 1–4 are included within this more general solution for their particular ξ restriction.

Taking the case 2 solution as an example,

$$\xi = C_w\rho_\infty\mu_\infty U_\infty\int_0^x\left(\frac{p_e}{p_\infty}\right)dx, \qquad G \approx \xi^{\frac{1}{2}}, \qquad p_e \approx \xi^N \tag{10}$$

Differentiating the relationship for ξ gives

$$\frac{d\xi}{dx} = C_w\rho_\infty\mu_\infty U_\infty\frac{p_e}{p_\infty} \tag{11}$$

which can be written from Eqs. (10) as

$$\frac{d\xi}{dx} \approx \xi^N \tag{12}$$

That is,

$$\begin{aligned} \xi &= (A + Bx)^{1/(1-N)} \qquad N \neq 1 \\ &= e^{Ax} \qquad N = 1 \end{aligned} \tag{13}$$

Substituting this value of ξ back into Eq. (10) gives

$$\xi = (A + Bx)^{1/(1-N)}, \qquad G \approx (A + Bx)^{\frac{1}{2}(1-N)}$$

$$p_e \approx G^{2N} \qquad N \neq 1$$

$$\xi = e^{Ax}, \qquad G \approx e^{Ax/2}, \qquad p_e \approx G^2 \qquad N = 1$$

which corresponds to Eqs. (8) and (9) when $N = \lambda$, as required. Note that this case contains within it a power law and an exponential solution for the particular ξ.

References

[1]Inger, G. R., "New Similarity Solutions for Hypersonic Boundary Layers with Applications to Inlet Flows," *AIAA Journal*, Vol. 33, No. 11, 1995, pp. 2080–2086.

[2]Mirels, H., "Hypersonic Flow over Slender Bodies with Power Law Shocks," *Advances in Applied Mechanics VII*, Academic, New York, 1962, pp. 1–54.

Reply by the Author to Jack Pike

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PART 1 of the Comment properly draws attention to the considerations involved in tailoring at $x = x_0$ an upstream boundary layer to the new similarly solution of Ref. 1. Since space limitations there did not permit a detailed discussion of these, it will be done here. In so doing, it is deemed sufficient for engineering purposes to match the skin friction and heat transfer across x_0; a further match of the displacement thickness is only significant in the rare practical application cases[2] where strong viscous-inviscid interaction effects are important (nevertheless, a treatment of it is included for the sake of completeness).

As the Comment observes, there are several arbitrary constants associated with the G function of Ref. 1 and, hence, with the resulting pressure and boundary-layer solution properties; these, together with x_0 and both the upstream wall temperature and pressure distribution history, are thus at our disposal to effect a match in the general situation. To fix ideas, consider the specific case 3 of an exponential pressure distribution (see Ref. 1 for nomenclature). Then the upstream boundary layer at $x = x_{\bar{0}}$ must match the similarity solution values

$$\frac{p_e(x_{\bar{0}})}{p_\infty} = K_{p_1}\exp(\beta x_0/L) \equiv K_\beta$$

with

$$C_f(x_{\bar{0}}) = \sqrt{\frac{2\beta K_\beta C_w^+}{R_{e_L}}}\cdot f_w''\left(\beta, g_w^+\right) \tag{1}$$

with

$$C_h(x_{\bar{0}}) = C_f(x_{\bar{0}})\cdot\frac{g_w'\left(\beta, g_w^+\right)}{f_w''\left(\beta, g_w^+\right)}$$

and

$$\delta^*(x_{\bar{0}}) = (\gamma - 1)M_\infty^2 L\sqrt{\frac{C_w^+}{2\beta K_\beta R_{e_L}}}\cdot I_g\left(\beta, g_w^+\right)$$

where $g_w^+ \equiv T_w(x_0^+)/T_0$ is the wall to total temperature ratio for the downstream similarity solution and L is an arbitrary reference length to be chosen advantageously. The left sides of these expressions pertain to an arbitrary nonsimilar arbitrary pressure gradient boundary-layer solution (of which a constant pressure ramp is only a special case), which can always be described by

$$C_f(x_{\bar{0}}) = \sqrt{\frac{2[p_e(x_{\bar{0}})/p_\infty]C_w^+}{Re_{x_0}}}\cdot f_w''(x_{\bar{0}}, g_{\bar{w}}) \tag{2}$$

with

$$C_h(x_{\bar{0}}) = C_f(x_{\bar{0}})\cdot\frac{g_w'(x_{\bar{0}}, g_{\bar{w}})}{f_w''(x_{\bar{0}}, g_{\bar{w}})}$$

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